International Journal of Theoretical Physics, Vol. 23, No. 12, 1984

Extended Feynman Formula for Forced Harmonic Oscillator

Bin Kang Cheng[1]

Departamento de Física, Universidade Federal do Paraná, 80.000 Curitiba, Caixa Postal 19.081, Brazil

Received October 27, 1983

Horvâthy's modification of Feynman's original method is generalized to the path integral formula of a forced harmonic oscillator. With this new formula the propagator of a harmonic oscillator with memory is evaluated exactly beyond and at caustics.

1. INTRODUCTION

For a forced harmonic oscillator the propagator has the form

$$K(b, a; T) = F(t_b, t_a)\exp\left[iS_{\mathrm{cl}}(b, a; T)/\hbar\right] \tag{1}$$

where $S_{\mathrm{cl}}(b, a; T)$ is the classical action and the path integral $F(t_b, t_a)$ is given by (Feynman and Hibbs, 1965, pp. 58–73; Schulman, 1981, pp. 31–41)

$$F(t_b, t_a) = \int_0^0 \exp\left[(im/2\hbar)\int_{t_a}^{t_b}(\dot{\eta}^2 - \omega^2\eta^2)\,dt\right] D\eta(t) \tag{2}$$

with $\eta(t_a) = \eta(t_b) = 0$. (1.1) is valid only for $T \equiv t_b - t_a < \tau/2$ ($\tau = 2\pi/\omega$), a half-period. Slightly modifying Feynman's original method, Horvâthy (1979) is able to evaluate (2) by including the Maslov correction (Arnold, 1967; Souriau, 1976) beyond caustics. Furthermore, he calculates (1) at caustics by using the integral form of Schrödinger's equation. In other

[1]Work supported in part by the Conselho Nacional de Desenvolvimento Cientifico e Tecnólogico (CNPq), Brazil.

words, we have

$$F(t_b, t_a) = \begin{cases} (m\omega/2\pi\hbar|\sin\omega T|)^{1/2}\exp\{(-i\pi/2)[1/2+\mathrm{Ent}(\omega T/\pi)]\}, & \omega T \neq k\pi \quad (3) \\ (m\omega/2\pi\hbar)\exp(-ik\pi/2), & \omega T = k\pi \quad (4) \end{cases}$$

and k is an integer. The Maslov correction in (3) has been observed in classical optics (Gouy, 1890) and in electron optics (Schulman, 1975) as well as in molecular (Miller, 1970) and nuclear (Levit *et al.*, 1974) scattering.

The purpose of the present paper is to study the classical action part of (1) beyond and at caustics for forced harmonic oscillator. With our new results we are able to evaluate exactly the propagator of a harmonic oscillator with memory, which has been treated by Cheng (1984) for real time and by Papadopoulos (1974), Maheshwari (1975), and Khandekar *et al.* (1983) for imaginary time, beyond and at caustics.

2. FORCED HARMONIC OSCILLATOR

For a forced harmonic oscillator the Larangian has the form

$$L(x, \dot{x}) = (m/2)[\dot{x}^2 - \omega^2 x^2] + fx \tag{5}$$

where ω and f are, respectively, the frequency and the constant external force. The classical action can then be written

$$S_{\mathrm{cl}}(b, a; T) = (m\omega/2\sin\omega T)\big[(x_b^2 + x_a^2)\cos\omega T - 2x_b x_a + 2A(T)(x_b + x_a) - B(T)\big] \tag{6}$$

with

$$A(T) = f(1-\cos\omega T)/m\omega^2 \qquad \text{and}$$

$$B(T) = f^2[2(1-\cos\omega T) - \omega T\sin\omega T]/m^2\omega^4 \tag{7}$$

As is well known, (6) is only valid for $T \neq k\tau/2$. Therefore, combining (1), (3), and (2), we obtain the propagator of forced harmonic oscillator beyond caustics

$$K(b,a;T)=(m\omega/2\pi\hbar|\sin\omega T|)^{1/2}\exp\{-(i\pi/2)[1/2+\mathrm{Ent}(\omega T/\pi)]\}$$
$$\times\exp\{(im\omega/2\hbar\sin\omega T)[(x_b^2+x_a^2)\cos\omega T$$
$$-2x_bx_a+2A(T)(x_b+x_a)-B(T)]\} \tag{8}$$

In order to obtain the propagator at caustics, we apply the following semigroup property

$$K(b,a;k\tau/2)=\int_{-\infty}^{\infty}K[b,c;(2k-1)\tau/4]\,K(c,a;\tau/4)\,dx_c \tag{9}$$

Substituting (8), respectively, for $T=\tau/4$ and $T=(2k-1)\tau/4$ into (9), we have

$$K(b,a;k\tau/2)=(m\omega/2\pi\hbar)\exp(-ik\pi/2)P(b,a) \tag{10}$$

where the classical action part of (9), $P(b,a)$ given by

$$P(b,a)=\int_{-\infty}^{\infty}\exp(-(i/\hbar)\{S_{\mathrm{cl}}[b,c;(2k-1)\tau/4]+S_{\mathrm{cl}}(c,a;\tau/4)\})\,dx_c$$
$$=(2\pi\hbar/m\omega)\exp\Big(-(im\omega/2\hbar)\{B(\tau/4)-(-1)^kB[(2k-1)\tau/4]\}\Big)$$
$$\times\exp\Big((im\omega/\hbar)\{A(\tau/4)x_a-(-1)^kA[(2k-1)\tau/4]x_b\}\Big)$$
$$\times\delta\Big([x_a-A(\tau/4)]-(-1)^k\{x_b-A[(2k-1)\tau/4]\}\Big) \tag{11}$$

after integration over x_c from $-\infty$ to ∞. With the help of (7) for $T=\tau/4$ and $T=(2k-1)\tau/4$, respectively, we finally obtain our principal result:

$$K(b,a;k\tau/2)=\exp(-ik\pi/2)\exp\Big((-if^2/\hbar m\omega^3)\{2[1-(-1)^k]-k\pi\}\Big)$$
$$\times\exp\{(if/\hbar\omega)[x_a-(-1)^kx_b]\}$$
$$\times\delta\{[x_a-(-1)^kx_b]-[1-(-1)^k](f/m\omega^2)\} \tag{12}$$

the propagator of forced harmonic oscillator at caustics. For harmonic oscillator case, $f=0$, (12) reduces to (3) of Horvàthy (1979) as we expect.

3. HARMONIC OSCILLATOR WITH MEMORY

For a harmonic oscillator with memory, studied by Cheng (1984), the Lagrangian has the following form:

$$L(x,\dot{x}) = (m/2)\left[\dot{x}^2 - \omega^2 x^2 + (2\omega^2/T)x\int_{t_a}^{t} x(t')\,dt'\right] \tag{13}$$

The propagator can then be expressed

$$K_m(b,a;T) = \int_a^b \exp\left[(im/2\hbar)\int_{t_a}^{t_b}(\dot{x}^2 - \omega^2 x^2)\,dt\right] \times \exp\left\{(im\omega^2/2\hbar T)\left[\int_{t_a}^{t_b} x(t)\,dt\right]^2\right\} Dx(t) \tag{14}$$

after integration by parts of the memory term in path integral. The difficult part of the path integral (14) is the last exponential functional, which involves off-diagonal terms. Introduce the following Gaussian integral (Papadopoulos, 1974):

$$\exp\left\{(im\omega^2/2\hbar T)\left[\int_{t_a}^{t_b} x(t)\,dt\right]^2\right\} = (im/2\pi\hbar T)^{1/2}\int_{-\infty}^{\infty}\exp\left[-(im/2\hbar T)u^2 + (im\omega/\hbar T)u\int_{t_a}^{t_b} x(t)\,dt\right]du \tag{15}$$

where u is an auxiliary variable. Substituting (15) into (14), we have

$$K_m(b,a;T) = (im/2\pi\hbar T)^{1/2}\int_{-\infty}^{\infty}\exp\left[-(im/2\hbar T)u^2\right]H(b,a;u)\,du \tag{16}$$

where the path integral

$$H(b,a;u) = \int_a^b \exp\left\{(im/2\hbar)\int_{t_a}^{t_b}\left[\dot{x}^2 - \omega^2 x^2 + (2\omega u/T)x\right]dt\right\} Dx(t) \tag{17}$$

is the propagator of a forced harmonic oscillator with frequency ω and with a constant external force $m\omega u/T$. Thus the memory term in (14) has been removed by first introducing an auxiliary variable u and than by integrating over u from $-\infty$ to ∞.

With the help of (7) for $f = m\omega u/T$, (8), (16), and (17), we obtain beyond caustics

$$\begin{aligned} &K_m(b, a; T) \\ &\quad = (m/2\pi\hbar T)^{1/2} \exp\{-(i\pi/2)[1/2 + \mathrm{Ent}(\omega T/\pi)]\} \\ &\quad\quad \times (\omega T/2|\sin(\omega T/2)|) \exp\left\{im\omega \cot(\omega T/2)(x_a - x_b)^2/4\hbar\right\} \\ &\quad\quad \times \begin{pmatrix} 1 \\ \exp(i\pi/2) \end{pmatrix} \end{aligned} \tag{18}$$

after lengthy but straightforward calculations. The last upper (lower) factor for $\tan(\omega T/2) > 0$ (< 0) is due to the integration over u from $-\infty$ to ∞ by using (14) of Horvâthy (1979). Substituting (12) with $f = m\omega u/T$ into (17) and than into (16), we finally obtain the propagator at caustics:

$$\begin{aligned} K_m(b, a; k\tau/2) = {} & (m/2\pi\hbar T)^{1/2} \exp[-i(2k-1)\pi/4] \\ & \times \exp\left[-im(x_a - x_b)^2/2\hbar T\right] \delta(x_a - x_b) \end{aligned} \tag{19}$$

for k even and

$$K_m(b, a; k\tau/2) = (m/2\pi\hbar T)^{1/2} (\omega T/2) \exp[-i(2k-1)\pi/4] \tag{20}$$

for k odd. Here we have used (14) of Horvâthy (1979) and the relation $\omega T = k\pi$ since $T = k\tau/2$ at caustics.

4. CONCLUSIONS

With our new results (18)–(20), the propagator (Cheng, 1984) for a charged particle in constant magnetic field, which is related to that of a harmonic oscillator with memory, can be evaluated beyond and at caustics. The new formulas (8) and (12) may also be generalized to evaluate the propagator beyond and at caustics of time-dependent forced harmonic oscillator with damping (Cheng, 1984). Work along these lines is in progress and will be published elsewhere.

REFERENCES

Arnold, V. (1967). *J. Func. Anal.*, **1**, 1–13.

Cheng, B. K. (1984). *Physics Scripta*, **29**, 351-352; *J. Math. Phys.*, **25**, 1804-1809.

Feynman, R. P., and Hibbs, A. R. (1965). *Quantum Mechanics and Path Intgral.* McGraw-Hill, New York.

Gouy, L. G. (1890). *Comptes Rendus Hebdomadaires des Seances de l'Academie des Sciences*, **110**, 1251.

Horvâthy, P. (1979). *Inter. J. Theor. Phys.*, **18**, 245–250.

Khandekar, D. C., Lawande, S. V., and Bhagwat, K. V. (1983). *Phys. Lett.*, **93A**, 167–172.

Levit, S., Smilansky, U., and Pèlte, D. (1974). *Phys. Lett.*, **53B**, 39.

Maheshwari, A. (1975). *J. Phys. A, Math., Nucl. Gen.*, **8**, 1019–1020.

Miller, W. H. (1970). *J. Chem. Phys.*, **53**, 1949, 3965.

Papadopoulos, G. J. (1974). *J. Phys. A, Math., Nucl. Gen.*, **7**, 183–185.

Schulman, L. S. (1981). *Techniques and Applications of Path Integrations.* John Wiley & Sons, New York.

Schulman, L. S. (1975). In *Functional Integrations and Its Applications*, A. M. Arthurs, ed. Clarendon Press, Oxford.

Souriau, J. M. (1976). Construction explicite de l'indice de Maslov. Applications, in *Group Theoretical Methods in Physics, IVth International Colloquium, Nijmegen (1975)*, A. Janner, ed. Springer Lecture Notes in Physics, No. 50. Springer, Berlin.